

On certain analytical solutions for viscoelastic half-space

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THE CHARACTERISTICS of boundary value problems in visco-elastic half-space are investigated. In particular the relations between individual boundary problems have been found and used in evaluation of new solutions. According to presented solutions an analytical treatment for ponderable half-space and for certain kinematic boundary conditions is included.

Omówiono własności zadań brzegowych w półprzestrzeni lepko-sprężystej. W szczególności udało się znaleźć relacje jakie łączą poszczególne zadania brzegowe i wykorzystać je do konstrukcji nowych rozwiązań. Na tej drodze podano analityczne rozwiązania dla ważkiej półprzestrzeni oraz dla pewnych kinematycznych warunków brzegowych.

Обсуждены свойства краевых задач в вязкоупругом полупространстве. В частности удалось найти зависимости объединяющие отдельные краевые задачи и использовать их для построения новых решений. На этом пути даются аналитические решения для весомого полупространства и для некоторых кинематических граничных условий.

1. Introduction

THE SOLUTIONS of boundary value problems for viscoelastic half-space already belong to classical problems. Many results connected with this subject are included in monographs on rheology, for example see [1, 2, 3, 5, 6]. It seems however that here not all problems have been widely explored, suffice it to mention the action of punch on viscoelastic half-space. This paper is concerned with a problem which has received less attention. In this study the properties of the solution of boundary value problems in viscoelastic half-space will be thoroughly discussed. It appears that the classic solution of the Boussinesq problem may be fruitfully used in constructing the closed-form solution for different boundary value problems of the half-space. Similar conclusions also hold for the Cerutti solution. In each of these cases new analytical solutions for several practically important problems may be obtained. We shall particularly dwell on only one solution connected with the stress analysis of deforming rock in the immediate neighbourhood of the structure. Such a solution is presented in [4].

2. The Boussinesq problem

We shall present first the solution of the Boussinesq problem in a half-space which is parametrized by the system of coordinates (x_1, x_2, x_3) and bounded by the plane $x_3 = 0$. This solution will be used later for constructing different solutions of boundary value problems in a viscoelastic half-space. The solution of the Boussinesq problem when

the concentration force P is directed along the axis x_3 , is given by the formulae (comp. [8], [9])

$$(2.1) \quad \begin{aligned} U_1(x_i, t) &= \frac{1}{4\pi} \left(\frac{x_1 x_3}{\mu r^3} - \frac{1}{\lambda + \mu} \frac{x_1}{r(x_3 + r)} \right) P_3(t), \\ U_2(x_i, t) &= \frac{1}{4\pi} \left(\frac{x_2 x_3}{\mu r^3} - \frac{1}{\lambda + \mu} \frac{x_2}{r(x_3 + r)} \right) P_3(t), \\ U_3(x_i, t) &= \frac{1}{4\pi} \left(\frac{x_3 x_3}{\mu r^3} - \frac{\lambda + 2\mu}{\lambda + \mu} \frac{1}{r} \right) P_3(t), \end{aligned}$$

where λ, μ are Lamé constants, $r^2 = x_1^2 + x_2^2 + x_3^2$ and t is the time.

In the viscoelastic problem in agreement with a viscoelastic analogue the constants μ and λ should be replaced by the correspondingly chosen functions of time $\dot{\mu}(t)$ and $\dot{\lambda}(t)$. Thus we have

$$(2.2) \quad \begin{aligned} \dot{U}_1(x_i, t) &= \frac{1}{4\pi} \left(\dot{\mu}^{-1} \frac{x_1 x_3}{r^3} - (\dot{\lambda} + \dot{\mu})^{-1} \frac{x_1}{r(x_3 + r)} \right) * \delta(t), \\ \dot{U}_2(x_i, t) &= \frac{1}{4\pi} \left(\dot{\mu}^{-1} \frac{x_2 x_3}{r^3} - (\dot{\lambda} + \dot{\mu})^{-1} \frac{x_2}{r(x_3 + r)} \right) * \delta(t), \\ \dot{U}_3(x_i, t) &= \frac{1}{4\pi} \left(\dot{\mu}^{-1} \frac{x_3 x_3}{r^3} - (\dot{\lambda} + \dot{\mu})^{-1} * (\dot{\lambda} + 2\dot{\mu}) \frac{1}{r} \right) * \delta(t), \\ \lambda * \lambda^{-1} &= \delta, \quad (\dot{\lambda} + \dot{\mu})^{-1} * (\dot{\lambda} + \dot{\mu}) = \delta, \quad P_3 = 1 \delta(t) \delta(x_1) \delta(x_2) \delta(x_3), \end{aligned}$$

where $\delta(t)$ is Dirac's distribution and the symbol $*$ denotes convolution.

For the load P prescribed on the simply connected closed domain the displacements in a half-space are given by the formulae

$$(2.3) \quad \begin{aligned} \hat{U}_1(x_i, t) &= \int_{\Gamma} \frac{1}{4\pi} \left(\dot{\mu}^{-1} \frac{(x_1 - x'_1) x_3}{r^3} - (\dot{\lambda} + \dot{\mu})^{-1} \frac{(x_1 - x'_1)}{r(x_3 + r)} \right) * P_3(x'_L, t) d\Gamma, \\ \hat{U}_2(x_i, t) &= \int_{\Gamma} \frac{1}{4\pi} \left(\dot{\mu}^{-1} \frac{(x_2 - x'_2) x_3}{r^3} - (\dot{\lambda} + \dot{\mu})^{-1} \frac{(x_2 - x'_2)}{r(x_3 + r)} \right) * P_3(x'_L, t) d\Gamma, \\ \hat{U}_3(x_i, t) &= \int_{\Gamma} \frac{1}{4\pi} \left(\dot{\mu}^{-1} \frac{x_3 x_3}{r^3} - (\dot{\lambda} + \dot{\mu})^{-1} * (\dot{\lambda} + 2\dot{\mu}) \frac{1}{r} \right) * P_3(x'_L, t) d\Gamma, \\ d\Gamma &= dx'_1 dx'_2, \quad L = 1, 2, \quad r^2 = (x_1 - x'_1)^2 + (x_2 - x'_2)^2 + x_3^2. \end{aligned}$$

The solutions (2.2) and (2.3) will be useful further on in evaluating new closed-form solutions for boundary value problems in viscoelastic half-space.

3. Properties of the boundary value problems of viscoelastic half-space

The viscoelastic displacement equations, the properties of which are the aim of our investigation, assume the following form:

$$(3.1) \quad \mu * dU_{i,jj} + (\lambda + \mu) * dU_{j,ji} + \rho X_i = 0, \quad i, j = 1, 2, 3.$$

Here X_i is a mass force, U_i denotes the displacement vector and λ and μ are material functions. In thermoelastic problems (uncoupled) the mass force ρX_i should be replaced by the expressions $\rho X_i - \gamma * \Theta_{,i}$, $\Theta = T - T_0$ where $\gamma = \alpha_T(3\lambda + 2\mu)$, α_T is a coefficient of thermal expansion, T is the known temperature of the medium and T_0 denotes the temperature in a natural state.

The characteristics of Eqs. (3.1) will be examined by analyzing the Fourier and Laplace-Carson transforms of the governing equations. The Fourier transforms are determined by the relations

$$(3.2) \quad \mathcal{F}[f(x_i, t)] = (2\pi)^{-\frac{3}{2}} \int_{-\infty}^{+\infty} \int \int f(x_i, t) e^{-i(\alpha_i x_i)} dx_1 dx_2 dx_3,$$

$$(3.3) \quad \mathcal{F}^{-1}[\tilde{f}(\alpha_i, t)] = (2\pi)^{-\frac{3}{2}} \int_{-\infty}^{+\infty} \int \int \tilde{f}(\alpha_i, t) e^{+i(\alpha_i x_i)} d\alpha_1 d\alpha_2 d\alpha_3.$$

Further, the following formulae for transforms of functions and their derivatives will be particularly useful,

$$(3.4) \quad \begin{aligned} \mathcal{F}[U_{i,j}(x_i, t)] &= i\alpha_j \tilde{U}_i(\alpha_i, t), \\ \mathcal{F}[U_{i,lj}(x_i, t)] &= -\alpha_l \alpha_j \tilde{U}_i(\alpha_i, t), \\ \mathcal{F}[U_{i,jj}(x_i, t)] &= -\alpha_j \alpha_j \tilde{U}_i(\alpha_i, t) - i\alpha_3 \tilde{U}_i(\alpha_L, 0^+, t) - \tilde{U}_{i,3}(\alpha_L, 0^+, t), \\ \mathcal{F}[U_{i,3j}(x_i, t)] &= -\alpha_3 \alpha_j \tilde{U}_i(\alpha_i, t) - i\alpha_j \tilde{U}_i(\alpha_L, 0^+, t), \\ \mathcal{F}[U_{i,33}(x_i, t)] &= -\alpha_3 \alpha_3 \tilde{U}_i(\alpha_i, t) - i\alpha_3 \tilde{U}_i(\alpha_L, 0^+, t) - \tilde{U}_{i,3}(\alpha_L, 0^+, t) \\ &I, J, K, L = 1, 2, \quad i, j, k, l = 1, 2, 3. \end{aligned}$$

The Laplace-Carson transform is determined by the relations

$$(3.5) \quad \mathcal{L}[f(\dots, t)] = p \int_0^{\infty} f(\dots, t) e^{-pt} dt,$$

$$(3.6) \quad \mathcal{L}^{-1}[\tilde{f}(\dots, p)] = \frac{1}{2\pi} \int_{x-i\infty}^{x+i\infty} f(\dots, p) e^{pt} dp, \quad \mathcal{L} \circ \mathcal{F}[f] = \hat{f}.$$

After performing the transforms (3.2), (3.5) and using the relations (3.4), we obtain the following matrix equations:

$$(3.7) \quad \mathbf{A}\hat{\mathbf{U}} = -\hat{\mathbf{X}} - \bar{\gamma}\hat{\mathbf{T}}, \quad \hat{\mathbf{T}} \equiv \mathcal{F}[\Theta_{,i}],$$

where

$$\mathbf{A} = \begin{bmatrix} -\bar{\mu}\alpha_j\alpha_j - (\bar{\mu} + \bar{\lambda})\alpha_1^2; & -(\bar{\lambda} + \bar{\mu})\alpha_1\alpha_2; & -(\bar{\lambda} + \bar{\mu})\alpha_1\alpha_3 \\ -(\bar{\lambda} + \bar{\mu})\alpha_2\alpha_1; & -\bar{\mu}\alpha_j\alpha_j - (\bar{\mu} + \bar{\lambda})\alpha_2^2; & -(\bar{\lambda} + \bar{\mu})\alpha_2\alpha_3 \\ -(\bar{\lambda} + \bar{\mu})\alpha_3\alpha_1; & -(\bar{\lambda} + \bar{\mu})\alpha_3\alpha_2; & -\bar{\mu}\alpha_j\alpha_j - (\bar{\lambda} + \bar{\mu})\alpha_3^2 \end{bmatrix}, \quad \hat{\mathbf{U}} = \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \\ \hat{U}_3 \end{bmatrix},$$

$$\begin{aligned} \hat{X}_1 &= \hat{X}_1(\alpha_j, p) - \bar{\mu}\hat{U}_1(\alpha_L, 0^+, p) i\alpha_3 - \bar{\mu}\hat{U}_{1,3}(\alpha_L, 0^+, p) - i\alpha_1 \bar{\mu}\hat{U}_3(\alpha_L, 0^+, p) \\ &\quad - \bar{\lambda}\hat{U}_3(\alpha_L, 0^+, p) i\alpha_1, \end{aligned}$$

$$(3.8) \quad \hat{X}_2 = \hat{X}_2(\alpha_j, p) - \bar{\mu} \hat{U}_2(\alpha_L, 0^+, p) i\alpha_3 - \bar{\mu} \hat{U}_{2,3}(\alpha_L, 0^+, p) - \bar{\mu} i\alpha_2 \hat{U}_3(\alpha_L, 0^+, p) \\ - \bar{\lambda} \hat{U}_3(\alpha_L, 0^+, p) i\alpha_2, \\ \hat{X}_3 = \hat{X}_3(\alpha_j, p) - \bar{\mu} \hat{U}_3(\alpha_L, 0^+, p) i\alpha_3 - \bar{\mu} \hat{U}_{3,3}(\alpha_L, 0^+, p) - \bar{\mu} \hat{U}_{3,3}(\alpha_L, 0^+, p) \\ - (\bar{\lambda} + \bar{\mu}) i\alpha_3 \hat{U}_3(\alpha_L, 0^+, p) - \bar{\lambda} [i\alpha_1 \hat{U}_1(\alpha_L, 0^+, p) + i\alpha_2 \hat{U}_2(\alpha_L, 0^+, p) \\ + \hat{U}_{3,3}(\alpha_L, 0^+, p)] - \bar{\lambda} i\alpha_1 \hat{U}_1(\alpha_L, 0^+, p) - \bar{\lambda} i\alpha_2 \hat{U}_2(\alpha_L, 0^+, p).$$

Then, collecting in Eq. (3.7) the corresponding terms and using constitutive equations (transformed) the following equation is obtained:

$$(3.9) \quad \mathbf{A}\hat{\mathbf{U}} = \mathbf{Y}, \quad \mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \mathbf{Y}_4 + \mathbf{Y}_5,$$

where

$$(3.10) \quad \mathbf{Y}_1 = \begin{bmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \hat{X}_3 \end{bmatrix}, \quad \mathbf{Y}_2 = \begin{bmatrix} \bar{\mu} \hat{U}_1(\alpha_L, 0^+, p) \\ \bar{\mu} \hat{U}_2(\alpha_L, 0^+, p) \\ (2\bar{\mu} + \bar{\lambda}) \hat{U}_3(\alpha_L, 0^+, p) \end{bmatrix} i\alpha_3, \quad \mathbf{Y}_3 = \begin{bmatrix} -\hat{p}_1(\alpha_L, 0^+, p) \\ -\hat{p}_2(\alpha_L, 0^+, p) \\ -\hat{p}_3(\alpha_L, 0^+, p) \end{bmatrix}, \\ \mathbf{Y}_4 = \begin{bmatrix} -\bar{\lambda} i\alpha_1 \hat{U}_3(\alpha_L, 0^+, p) \\ -\bar{\lambda} i\alpha_2 \hat{U}_3(\alpha_L, 0^+, p) \\ -\bar{\lambda} \hat{\varepsilon}_{kk}(\alpha_L, 0^+, p) \end{bmatrix}, \quad \mathbf{Y}_5 = \begin{bmatrix} i\alpha_1 \hat{\Theta}(\alpha_j, p) \\ i\alpha_2 \hat{\Theta}(\alpha_j, p) \\ i\alpha_3 \hat{\Theta} - \hat{\Theta}(\alpha_L, 0^+, p) \end{bmatrix} \bar{\gamma}, \quad \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22}.$$

Note that the equality of vectors \mathbf{Y}_3 and \mathbf{Y}_4 i.e.

$$[0, 0, -\hat{p}_3(\alpha_L, 0^+, p)]^T = [0, 0, -\bar{\lambda} \hat{\varepsilon}_{kk}(\alpha_L, 0^+, p)]^T,$$

implies the identity of the displacement states caused by the normal force p_3 and compression ε_{kk} acting on the same boundary of the half-space. This property was used in Paper [4] to evaluate the displacement states in the neighbourhood of the contact of the structure with the deforming rock. Due to the linearity of Eq. [3.9] one has

$$(3.11) \quad (\mathbf{A}\hat{\mathbf{U}} = \mathbf{Y}) \Rightarrow (\mathbf{A}\varphi(\alpha_3)\hat{\mathbf{U}} = \varphi(\alpha_3)\mathbf{Y})$$

i.e. the solution $\varphi(\alpha_3)\hat{\mathbf{U}}$ corresponds to the vector $\varphi(\alpha_3)\mathbf{Y}$.

The characteristics of Eq. (3.1) presented above will be used in deriving the solution for two boundary value problems of the half-space. The main points are: the Boussinesq solution, the relations (3.10) and the properties of transforms of the displacement equations (3.11).

3.1. Action of the mass force

If the mass force vector in a half-space is presented as $X_i = [0, 0, \varrho X_3(x_L, t)\varphi(x_3)]^T$ and the displacement vector \hat{U}_i corresponds to the normal force $p_3(x_1, x_2, 0^+, t) = \varrho X_3(x_1, x_2, t)$ (compare Eqs. (2.2) and (2.3)), then the displacement state in the ponderable half-space is determined by the relation

$$(3.12) \quad U_i = \int_0^{x_3} \hat{U}_i(x_1, x_2, y_3 - x_3, t)\varphi(y_3)dy_3.$$

3.2. Action of the displacement $U_3(x_1, x_2, 0^+, t)$

In this case the components of the displacement vector in the medium cannot be obtained immediately; one can only get a certain system of differential relations from which the vector U_i is evaluated. If the state of the displacements \hat{U}_i corresponds to the vector $p_3(x_1, x_2, 0^+, t)$ and the following equalities hold,

$$(3.13) \quad -(\lambda + 2\mu) * dU_3(x_1, x_2, 0^+, t) = -p_3(x_1, x_2, 0^+, t)$$

then, the displacement state in a half-space caused by the field $U_3(x_1, x_2, 0^+, t)$ is determined by the equations

$$(3.14) \quad U_i = \frac{d}{dx_3} \int_{\Gamma} [\hat{U}_i(x_L - x'_L, x_3, t) * (\lambda + 2\mu) * U_3(x'_L, 0^+, t)] d\Gamma + \\ \int_{\Gamma} \hat{U}_i(x_L - x'_L, 0^+, t) * (\lambda + 2\mu) * U_3(x'_L, 0^+, t) d\Gamma.$$

A similar procedure may be applied to the temperature field. However, in this case the Cerutti solution should also be used.

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