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Free molecular flow over a slightly waved a wall

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A FREE-MOLECULAR flow over a slightly waved wall is considered. It is assumed that the ratio of the amplitude of the wave to its length is small. Also the angle between mean velocity of particles falling on the wall and the normal to the surface is considered to be of the same order. Assuming diffusive reflection the influence of waving of the wall on the distribution function and hydrodynamic quantities is examined. The hydrodynamic quantities are presented in a closed form and are expressed in terms of elementary functions or single quadratures.

THE PURPOSE of this paper is to analyse the free-molecular flow over a regularly undulated wall, which is of some interest in practical applications and may be helpful in analysing more involved configurations.

The mathematical difficulty connected with the concave geometry is avoided by assumming that the ratio of the amplitude of wall undulations to the wave length is much smaller then one ($\lambda \ll 1$).

The following assumptions are made for the present analysis:

a) The flow of a highly rarefied gas over an infinite wall described by $y = \lambda g(x)$ in Cartesian (x, y, z) coordinates is considered, $g(x)$ is a periodic function.

b) The wall is impermeable and nonabsorbing.

c) The molecules are reflected diffusively with the temperature equal to the wall temperature T_0.

d) The impinging molecules have a Maxwellian distribution with temperature, density and velocity equal respectively T_0, n_0, $\mathbf{v}$.

e) The average flow is two-dimensional, stationary and collisionless.

The Boltzmann equation describing the flow is:

$$c_1 \frac{\partial f}{\partial x} + c_2 \frac{\partial f}{\partial y} = 0 \tag{1}$$

with the boundary conditions

$$f(\mathbf{c}, \mathbf{x}_s) = \begin{cases} n(\mathbf{x}_s)\left(\dfrac{h}{\pi}\right)^{3/2} e^{-c^2}, & \mathbf{cn} > 0, \\ n_0\left(\dfrac{h}{\pi}\right)^{3/2} e^{-(\mathbf{c}-\mathbf{v})^2}, & \mathbf{cn} < 0, \quad \mathbf{c} \in \Omega, \end{cases} \tag{2}$$

where $\mathbf{x}_s = (\tau, \lambda g(\tau))$ is a point on the two-dimensional boundary line. Ω is the angle between the plane surfaces tangent to the boundary and passing through $\mathbf{x}_s$ as shown in Fig. 1. $\mathbf{n}$ is the vector normal to the boundary at $\mathbf{x}_s$, $\mathbf{c} = (c_1, c_2, c_3)$ is the molecular velocity of the particles measured in units of the "thermal velocity" $\dfrac{1}{\sqrt{h}} = \sqrt{\dfrac{2kT_0}{m}}$.

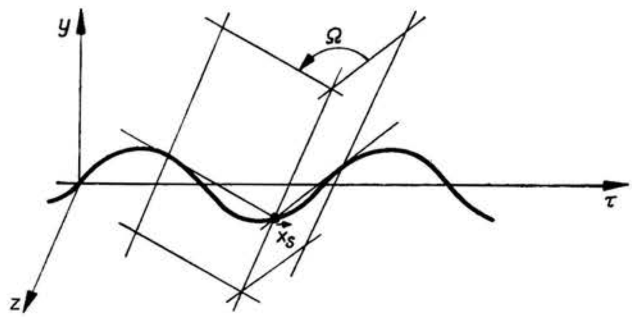

FIG. 1.

As appears from (1) the distribution function $f(\mathbf{c}, x, y)$ is constant along the characteristics and hence knowing its value at the boundary it can be determined at any point in the space considered.

The unknown function $n(\mathbf{x}_s)$ in (2) is determined from the condition of impermeability of the wall.

Introducing cylindrical coordinates

$$
\begin{aligned}
c_1 &= -\varrho\sin\theta,\\
c_2 &= \varrho\cos\theta,\\
c_3 &= c_3,\\
-\pi &< \theta \leqslant \pi
\end{aligned}
\tag{3}
$$

in the molecular velocity space connected with the geometry of the wall (Fig. 2), the equation for the number density at the wall is

$$
\mathcal{N}(\tau) = \mathcal{N}_\infty(\tau) + \int\limits_{\alpha(\tau)} \mathcal{N}(\tau_1)\frac{\partial}{\partial\tau_1}(\sin\theta)\,d\tau_1,
\tag{4}
$$

where $\mathcal{N}(\tau) = N(\mathbf{x}_s)$ is the number of particles reflected from a unit area in a unit of time at point $\mathbf{x}_s$, $\mathcal{N}_\infty(\tau)$ is the number of particles coming directly from infinity, $\alpha(\tau)$ is the set of values τ_1 such that the point $\mathbf{x}_{s1} = (\tau_1, \lambda g(\tau_1))$ is visible from the point $\mathbf{x}_s$, for a sinusoidal boundary $\alpha(\tau)$ is the segment whose endpoints a, b correspond to the tangents to the boundary line passing through $\mathbf{x}_s, (a, \lambda g(a)), (b, \lambda g(b))$, see Fig. 3.

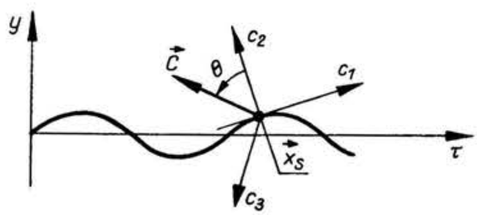

FIG. 2.

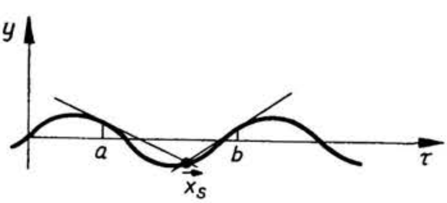

FIG. 3.

It can bee shown that the kernel of the integral in (4) is small $\frac{\partial}{\partial \tau_1}(\sin\theta) = 0(\lambda^2)$ and there from if $\mathcal{N}_\infty(\tau)$ is limited and $\lambda \ll 1$ the contribution of particles coming from other parts of the boundary is of second order and therefore the second term in the r.h.s. of (4) can be neglected with respect to the first r.h.s. term, hence

(5) $$\mathcal{N}(\tau) = \mathcal{N}_\infty(\tau).$$

The problem is reduced now to the determination of

(6) $$\mathcal{N}_\infty(\tau) = -\int_\Omega n_0 \frac{\mathbf{cn}}{\pi\sqrt{\pi h}} e^{-(\mathbf{c}-\mathbf{v})^2} d\mathbf{c}.$$

where the set Ω depends on the point $\mathbf{x}_s$.

Assume that the component of the bulk velocity $\mathbf{v}$ normal to the boundary line is small:

(7) $$|v\cos\psi| = |\mathbf{nv}| = |v_2 - \lambda g'(\tau) v_1| \ll 1.$$

The notation is explained in Fig. 4 an $g'(\tau) = \frac{dg}{d\tau}$. For this case (6) can be integrated and there follows:

(8) $$\mathcal{N}(\tau) = \mathcal{N}_\infty(\tau) = \frac{n_0}{2\sqrt{\pi h}}[1-\sqrt{\pi} v\cos\psi] = \frac{n_0}{2\sqrt{\pi h}}\left[1-\sqrt{\pi}\left(v_2 - \lambda v_1 g'(\tau)\right)\right].$$

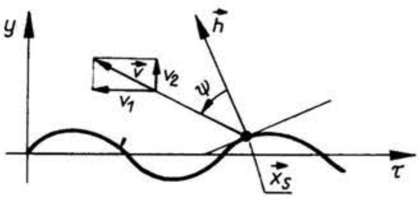

FIG. 4.

It is interesting to note that in the case $v \ll 1$ and hence $|v\cos\psi| = |\mathbf{nv}| = |v_2 - \lambda v_1 g'(\tau)| \ll 1$, $v_1 \ll 1$, $|\lambda v_1| \sim \lambda^2$

(9) $$\mathcal{N}(\tau) = \frac{n_0}{2\sqrt{\pi h}}[1 - v_2\sqrt{\pi}] = \text{const},$$

i.e., in this case, within the approximations made the flow is the same as over a plane wall.

In our further considerations we assume that $|v\cos\psi| \ll 1$ but that v_1 can be of order one.

From (2) and (8) there follows the distribution function:

(10) $$f(\mathbf{c}, x, y) = \begin{cases} n_0\left(\frac{h}{\pi}\right)^{3/2} e^{-(\mathbf{c}-\mathbf{v})^2}, & c_2 < 0, \\ n_0[1-\sqrt{\pi}(v_2 - \lambda v_1 g'(\tau))]\left(\frac{h}{\pi}\right)^{3/2} e^{-c^2}, & c_2 > 0, \end{cases}$$

where τ is determined from the condition that $(\tau, \lambda, g(\tau))$ is the intersection point of the boundary line and the characteristic passing through (x, y) i.e., the straight line parallel to the vector (c_1, c_2). Two examples of boundary lines will be considered: 1) thoot line and 2) a sinusoidal one.

1. The boundary line in the first case is described by

$$g(\tau) = \begin{cases} -a\tau - 1, & -\frac{2}{a} \leqslant \tau < 0, \\ \tau - 1, & 0 \leqslant \tau < 2 \end{cases} \tag{11}$$

which is periodic with the period $2+\frac{2}{a}$ and it is assumed that $a\lambda \leqslant 1$.

Denote the set of particle velocities passing through the point $A(x, y)$ from the boundary line with a negative slope by Ω_1, and from the part of the boundary with positive slope by Ω_2.

Then the distribition function is

$$f(\mathbf{c}, \mathbf{x}, y) = \begin{cases} n_0\left(\frac{h}{\pi}\right)^{3/2} e^{-(\mathbf{c}-\mathbf{v})^2}, & c_2 < 0, \\ n_0[1-\sqrt{\pi}(v_2 + a\lambda v_1)]\left(\frac{h}{\pi}\right)^{3/2} e^{-c^2}, & \mathbf{c} \in \Omega_1, \\ n_0[1-\sqrt{\pi}(v_2 - \lambda v_1)]\left(\frac{h}{\pi}\right)^{3/2} e^{-c^2}, & \mathbf{c} \in \Omega_2, \end{cases} \tag{12}$$

and the numerical density is:

for $-\frac{2}{a} < x < 0$

$$n(x, y) = n_0\left\{1 - v_2\left(\frac{1}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{2}\right) + \frac{v_1\lambda}{4}\left[(1-a)\sqrt{\pi} - \frac{2(1+a)}{\sqrt{\pi}} \times \right.\right.$$
$$\left.\left. \times \left(G(x, y) + \operatorname{arcctg}\left(-\frac{y+\lambda}{x}\right)\right)\right]\right\}; \tag{13}$$

for $0 < x < 2$

$$n(x, y) = n_0\left\{1 - v_2\left(\frac{1}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{2}\right) + \frac{v_1\lambda}{4}\left[(a+3)\sqrt{\pi} - \frac{2(1+a)}{\sqrt{\pi}} \times \right.\right.$$
$$\left.\left. \times \left(G(x, y) + \operatorname{arcctg}\left(-\frac{y+\lambda}{x}\right)\right)\right]\right\}, \tag{14}$$

where

$$G(x, y) = \sum_{n=1}^{i}\left[\operatorname{arcctg}\left(-\frac{y+\lambda}{x+2n+\frac{2n}{a}}\right) - \operatorname{arcctg}\left(-\frac{y-\lambda}{x+2(n-1)+\frac{2n}{a}}\right)\right]$$
$$+ \sum_{n=1}^{k}\left[\operatorname{arcctg}\left(-\frac{y+\lambda}{x-2n-\frac{2n}{a}}\right) - \operatorname{arcctg}\left(-\frac{y-\lambda}{x-2n-\frac{2(n-1)}{a}}\right)\right];$$

$$i = E\left[\frac{1}{2\left(1+\frac{1}{a}\right)}\left(\frac{y+\lambda}{\lambda}-x\right)\right],$$
(15)
$$k = E\left[\frac{1}{2\left(1+\frac{1}{a}\right)}\left(\frac{y+\lambda}{a\lambda}+x\right)\right].$$

2. The second case corresponds to the intersection of the plane $z = 0$ with the wall being a sinusoidal line. In this case the boundary line is given by the function $g(\tau) = \lambda \sin\tau$ and the corresponding distribution function is given by

(16)
$$f(\mathbf{c}, x, y) = \begin{cases} n_0\left(\frac{h}{\pi}\right)^{3/2} e^{-(\mathbf{c}-\mathbf{v})^2}, & c_2 < 0, \\ n_0[1-\sqrt{\pi}(v_2-\lambda v_1\cos\tau)]\left(\frac{h}{\pi}\right)^{3/2} e^{-c^2}, & c_2 > 0, \end{cases}$$

where τ must be determined from:

(17)
$$\lambda \sin\tau - y = (x-\tau)\operatorname{ctg}\theta.$$

It should be noted that for the calculation of the hydrodynamic magnitudes it is not necessary to solve (17) for τ; but (17) can be solved for θ and then perform the integration over the molecular velocities using θ as a variable instead of τ. Using the expressions (16) it can be shown that, qualitatively, within the approximations made, the influence of waviness on the main shape of the distribution function as compared with the distribution function of a plane wall is rather small.

The hydrodynamic magnitudes, number density and velocity components, are given by the following analytical expressions:

(18)
$$n(x, y) = n_0\left[1 - v_2\left(\frac{\sqrt{\pi}}{2}+\frac{1}{\sqrt{\pi}}\right) - \frac{\lambda v_1}{2\sqrt{\pi}}\int_{\alpha}^{\beta}\frac{\lambda[(x-\tau)\cos\tau+\sin\tau]-y}{(x-\tau)^2+(\lambda\sin\tau-y)^2}\cos\tau d\tau\right],$$

(19)
$$u_1(x, y) = \frac{n_0}{n}\left[\left(1-\frac{2v_2}{\sqrt{\pi}}\right)\frac{v_1}{2} - \frac{\lambda v_1}{4}\int_{\alpha}^{\beta}\frac{\lambda[(x-\tau)\cos\tau+\sin\tau]-y}{\left(\sqrt{(x-\tau)^2+(\lambda\sin\tau-y)^2}\right)^3}(x-\tau)\cos\tau d\tau\right];$$

(20)
$$u_2(x, y) = \frac{n_0}{n}\frac{\lambda v_1}{4}\int_{\alpha}^{\beta}\frac{\lambda[(x-\tau)\cos\tau+\sin\tau]-y}{\left(\sqrt{(x-\tau)^2+(\lambda\sin\tau-y)^2}\right)^3}(y-\lambda\sin\tau)\cos\tau d\tau,$$

α, β are the solutions with respect to τ of the equations

$$\lambda\sin\tau - y = \lambda(\tau - x), \qquad \sin\tau - y = -\lambda(\tau - x).$$

At large distances from the wall, for $1 < y < -2\ln\lambda$, neglecting terms of order λ^2, the above integrals can be calculated and there follows:

(21)
$$n(x, y) = n_0\left[1 - v_2\left(\frac{\sqrt{\pi}}{2}+\frac{1}{\sqrt{\pi}}\right)\right] + \frac{n_0\sqrt{\pi}\lambda v_1\cos x}{2}e^{-y},$$

$$(22) \qquad u_1(x,y) = \frac{v_1}{2}\left[1+\left(\frac{\sqrt{\pi}}{2}-\frac{1}{\sqrt{\pi}}\right)v_2\right]-\frac{\sqrt{\pi}\lambda v_1\cos x}{2}e^{-y}+\frac{\lambda v_1 y\sin x}{2}K_0(y)+\frac{\lambda^2 v_1\ln\lambda}{4},$$

$$(23) \qquad u_2 = -\frac{\lambda v_1\cos x}{2}\left(\sqrt{\pi}e^{-y}+y\frac{K_1(y)}{2}\right),$$

where $K_n(y)$ is the third kind modified Bessel function. When $y > -2\ln\lambda$ the perturbation terms are of order λ^2 and can be neglected. Using asymptotic development of the Bessel functions $K_0(y)$ and $K_1(y)$ there follows:

$$(24) \qquad u_1(x,y) = \frac{v_1}{2}\left[1+\left(\frac{\sqrt{\pi}}{2}-\frac{1}{\sqrt{\pi}}\right)v_2\right]-\frac{\sqrt{\pi}\lambda v_1}{2}\left[\cos x-\left(\frac{\sqrt{y}}{\sqrt{2}}-\frac{1}{8\sqrt{y}}\right)\sin x\right]e^{-y}+\frac{\lambda^2 v_1\ln\lambda}{4},$$

$$(25) \qquad u_2(x,y) = -\frac{\sqrt{\pi}\lambda v_1}{2}\cos x\left(1+\frac{\sqrt{y}}{2\sqrt{2}}+\frac{3}{16\sqrt{y}}\right)e^{-y}.$$

These approximations are valid for $a < y < -2\ln\lambda$ where a satisfies the equation $e^{-a}/a^{3/2} = \lambda$.

The integrals appearing in (18)–(20) have been calculated numerically close to the wall and the influence of the wall undulations on the flow is illustrated in the following figures. As expected the variation of the hydrodynamic magnitudes with x exhibits the same geometric features as the shape of the wall (Figs. 5, 6, 7).

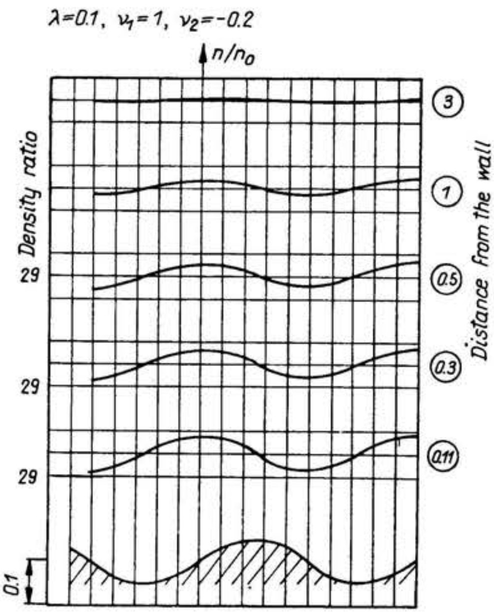

FIG. 5.

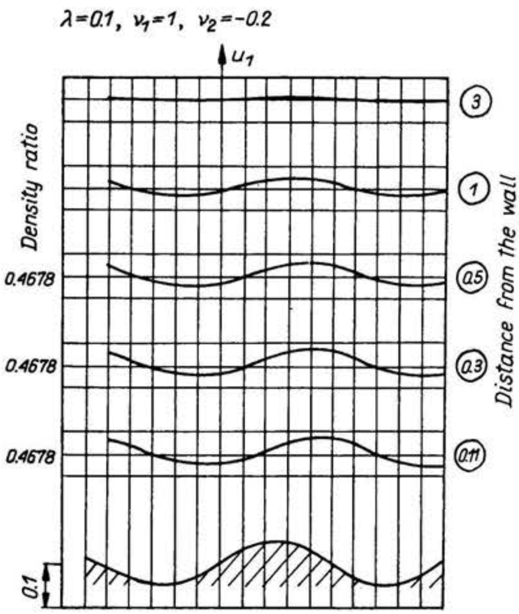

FIG. 6.

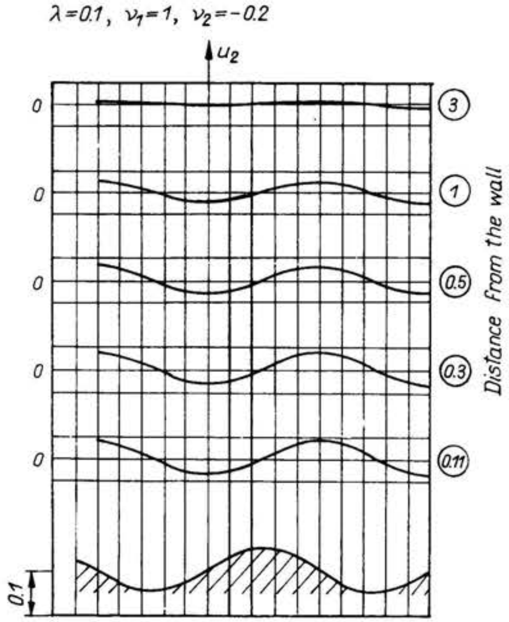

FIG. 7.

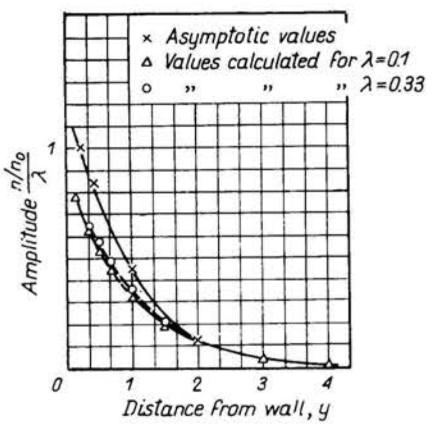

FIG. 8.

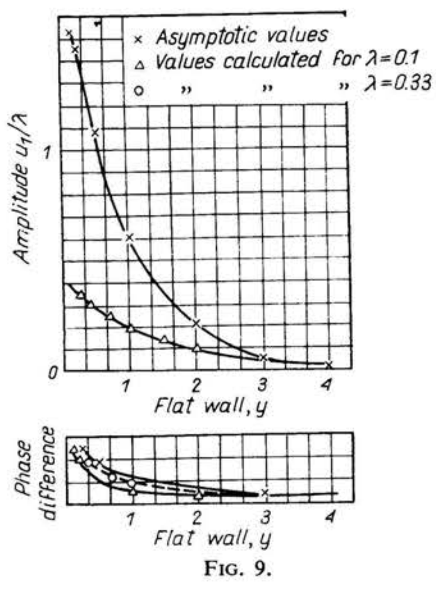

FIG. 9.

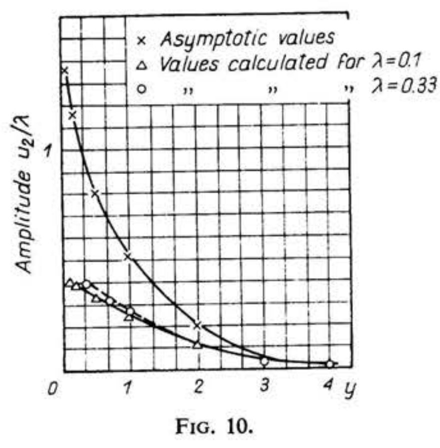

FIG. 10.

The amplitude of the perturbation of the density and velocity diminishes rapidly with the distance above the wall and at a distance above about three wave lengths decays exponentially according to the asymptotic exponential law (Figs. 8–10). Phase variations appear only for u_1 (the x component of the velocity) as shown in Fig. 9.

From the asymptotic formulae it appears that the amplitude of the variations of $n(x, y)$, $u_1(x, y)$, $u_2(x, y)$ with the distances from the wall is proportional to the amplitude of the wall undulations λ but the numerical calculations indicate for $n(x, y)$ and $u_2(x, y)$ a non-linear behaviour at distances from the wall below 3 wave lengths.

From (18)–(20) it can be noticed that within the range of validity of the assumptions made the perturbations due to wall shape are proportional to v_1, and independent from v_2. The volocity component normal to the wall v_2 produces only a constant correction, the same as in the case of a flat wall.

The solution of a similar problem for $u_2 = 0$ in continuum incompressible flow (Mach number $\mathsf{M} = 0$) gives for the average velocity a perturbation of the form:

$$\Delta u_1 = \frac{1}{2} v_1 \lambda e^{-y} \sin x,$$

$$\Delta u_2 = \frac{1}{2} v_1 \lambda e^{-y} \cos x,$$

which shows a marked similarity with the asymptotic relations particularly for the u_2 component.

However, the solutions obtained for the continuum model results for $\mathsf{M} > 1$, i.e.,

$$(\Delta u_1)_{\text{cont.}} = \frac{v_1 \lambda}{2\beta} \cos(x - \beta y),$$

$$(\Delta u_2)_{\text{cont.}} = \frac{v_1 \lambda}{2} \cos(x - \beta y)$$

$$\beta = \mathsf{M}^2 - 1$$

do not show an exponential attenuation. This difference is of course due to the approximations made in both cases.

The asymptotic solutions also indicate that the perturbation term in u_1 due to the wall waviness contains besides the term proportional to $\exp(-y)$ a term independent of y equal to $\frac{\lambda^2 v_1}{4} \ln \lambda$ which indicates that the wall undulations introduce a small reduction of the average flow velocity as compared with the flat wall case.

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