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BRIEF NOTES

Drag of a flat plate in a slip flow
A bivariational approach

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THE AERODYNAMIC drag coefficient of a finite flat plate, in a uniform flow at low Reynolds numbers, is evaluated in the presence of velocity jump conditions at the wall. Complementary bivariational principles are applied and the results are in excellent agreement with the values obtained through more difficult analytic approaches.

1. Introduction and basic equations

THE FLOW field of a slightly rarefied gas past an aligned finite flat plate, at a low Reynolds number, can be adequately described by the Oseen equation with a velocity slip condition at the wall. In a recent paper [1] a mathematical model was introduced, where the plate is represented by a distribution of Oseenlets, so that the Oseen equation is given in an integral form as

$$(1.1)\qquad 1+\left(\frac{1}{2\pi}\right)\int_0^1\left[\ln|x-t|-\ln\frac{4}{R}+\gamma-1\right]f(t)dt = k\cdot f(x),\qquad (0<x<1).$$

In Eq. (1.1) γ is Euler's constant, $R = lU/\nu$ is the Reynolds number and f is the shear stress at the wall. Furthermore, x is the coordinate parallel to the wall and normalized with respect to the length of the plate l, while the uniform velocity of the free stream is U and ν is the kinematic viscosity.

If in the x-direction u is the velocity perturbation, normalized by U, the slip condition at the surface $y = 0$ is expressed by

$$(1.2)\qquad kf(x) = k|\partial u/\partial y|_{y=0} = 1+u,$$

where k is the slip coefficient. On the other hand, the drag coefficient C_D is defined as

$$(1.3)\qquad C_D = \frac{4}{R}\int_0^1 \frac{\partial u}{\partial y}\bigg|_{y=0} dx = \frac{4}{R}\int_0^1 f(x)dx.$$

Introducing the condition (1.2) eliminates nonphysical singularities of the shear stress on the plate (see for instance Ref. [2] for a valuable introduction of the question).

While in [1] a solution of the problem was sought by approximately evaluating $f(x)$ and then calculating the drag coefficient, in this paper the determination of C_D is carried out directly by means of a bivariational approach, following the theory in [3].

To this purpose, Eq. (1.1) is first re-written in the form

$$\eta_0(x) = \eta(x)+h\int_0^1 K(x,\xi)\eta(\xi)d\xi \tag{1.4}$$

with $\eta_0(x) = 1$; $h = 1/2\pi k > 0$;

$$K(x,\xi) = \ln\frac{4}{R} - \gamma + 1 - \ln|x-\xi|. \tag{1.5}$$

Let $<,>$ indicate the inner product

$$\langle u, v\rangle = \int_0^1 u(x)v(x)dx, \quad u, v \in \mathscr{H}$$

in the Hilbert space $\mathscr{H}$ and let $T: \mathscr{H} \to \mathscr{H}$ be the integral operator defined by

$$Tu = u(x)+h\int_0^1 K(x,\xi)u(\xi)d\xi. \tag{1.6}$$

The problem expressed by Eqs. (1.3) can then be given the form

$$\eta_0 = T\eta, \qquad C_D = \frac{4}{kR}\langle \eta, \eta_0\rangle. \tag{1.7}$$

2. Complementary variational principles for C_D

COLE and PACK [3] furnished some complementary (or dual variational) principples for $<\eta, \eta_0>$, where $\eta_0 = T\eta$, which can be applied to the problem under investigatition provided that some properties of the operator T are verified. In particular,

1. *T is self-adjoint in $\mathscr{H}$*; i.e.

$$\langle u, Tv\rangle = \langle Tu, v\rangle. \tag{2.1}$$

This follows from the symmetry of the kernel (1.5) with respect to the variables (x, ξ).

2. *The relation*

$$0 < \langle u,u\rangle \leqslant \langle u, Tu\rangle \leqslant \langle u, \tau(x)u\rangle < m\langle u,u\rangle, \quad \forall u \in \mathscr{H} \tag{2.2}$$

holds, where the function $\tau(x)$ and the quantity m are defined by

$$\tau(x) = T1 = 1+h\int_0^1 K(x,\xi)d\xi, \tag{2.3}$$

$$m = \sup_{x\in[0,1]} \tau(x). \tag{2.4}$$

The expression (2.2) can be deduced from the relation

$$\langle u, Ku\rangle \leqslant \langle u^2, K1\rangle$$

for $K(x, \xi) \geqslant 0$, $x, \xi \in [0, 1]$.

Note that $K(x, \xi) \geqslant 0$ corresponds to the following limitation on the Reynolds number: $R < 4 \cdot \exp(1-\gamma) \simeq 6.10$, which is fully acceptable as the basic equation holds for small R. After verifying the two properties indicated above, the following four functionals are considered for the operator T, namely:

$$L_0(\phi) = \langle \phi, 2\eta_0 - T\phi \rangle, \tag{2.5}$$

$$L_1(\phi) = L_0 + m^{-1}\langle \eta_0 - T\phi, \eta_0 - T\phi \rangle, \tag{2.6}$$

$$L_2(\phi) = L_0 + \langle \eta_0 - T\phi, \tau^{-1}(\eta_0 - T\phi) \rangle, \tag{2.7}$$

$$U(\phi) = L_0 + \langle \eta_0 - T\phi, \eta_0 - T\phi \rangle \tag{2.8}$$

which satisfy the fundamental unequality([1])

$$L_0(\phi) \leqslant L_1(\phi) \leqslant L_2(\phi) \leqslant C_D \leqslant U(\phi) \tag{2.9}$$

for some trial functions $\phi \in \mathscr{H}$.

A proper choice of the trial functions appears to be

$$\phi_N = \sum_{j=0}^{N} \alpha_j P_{2j}(x), \tag{2.10}$$

where the P_{2j} are Legendre polynomials which, apart from their orthogonality property, show recurrence relations which are particularly suitable for the actual computations.

In Eq. (2.10) the constants α_k are obtained by substituting the trial functions in the functional and, subsequently, by determining the maximum (or the minimum) of the functional with respect to α_k.

The functionals (2.5) and (2.6) can be obtained from the single functional

$$W(\phi, \psi) = \langle \eta_0, \psi\eta_0 \rangle - \langle 2\eta_0, V\phi \rangle + \langle T\phi, V\phi \rangle, \tag{2.11}$$

where V is the self-adjoint operator defined as

$$V = \psi T - I$$

and I is the identity operator. In fact

$$L_0(\phi) = W(\phi, 0) : V = -I, \tag{2.12}$$

$$L_1(\phi) = W(\phi, m^{-1}) : V = m^{-1}T - I, \tag{2.13}$$

$$L_2(\phi) = W(\phi, \tau^{-1}) : V = \tau^{-1}T - I, \tag{2.14}$$

$$U(\phi) = W(\phi, 1) : V = T - I. \tag{2.15}$$

The operator V is negative definite in Eqs. (2.12)–(2.14) and is positive definite in Eq. (2.15). When $\phi_0 = \alpha_0$ is substituted into Eq. (2.11) one has

$$W(\phi_0, \psi) = \langle \eta_0, \psi\eta_0 \rangle - \alpha_0 \langle 2\eta_0, V1 \rangle + \alpha_0 \langle T1, V1 \rangle.$$

Substituting $\phi_1 = \alpha_0 + \alpha_1 P_2$ into Eq. (2.11) yields

$$W(\phi_1, \psi) = W(\phi_0, \psi) - \alpha_1 [\langle 2 - T\phi_0, VP_2 \rangle - \langle TP_2, V(\phi_0 + \alpha_1 P_2) \rangle]$$

([1]) This corresponds to seeking the functionals $L(\phi)$ and $U(\phi)$ of an approximating function ϕ such that

$$L(\phi) \leq \max_{\phi} L(\phi) = \langle \eta, \eta_0 \rangle = \min_{\phi} U(\phi) \leqslant U(\phi).$$

while $\phi_2 = \alpha_0 + \alpha_1 P_2 + \alpha_2 P_4$ gives

$$W(\phi_2, \psi) = W(\phi_1, \psi) - \alpha_2[\langle 2 - T\phi 1, VP_4\rangle - \langle TP_4, V(\phi_1 + \alpha_2 P_4)\rangle].$$

Optimizing

A) $W(\phi_0, \psi)$ with respect to α_0,

B) $W(\phi_1, \psi)$ with respect to α_0, α_1,

C) $W(\phi_2, \psi)$ with respect to $\alpha_0, \alpha_1, \alpha_2$

leads to

A) $\alpha_0 = \dfrac{\langle \eta_0, V1\rangle}{\langle T1, V1\rangle}$,

B) $\langle T1, V1\rangle\alpha_0 + \langle T1, VP_2\rangle\alpha_1 = \langle 1, V1\rangle$,
$\langle T1, VP_2\rangle\alpha_0 + \langle TP_2, VP_2\rangle\alpha_1 = \langle 1, VP_2\rangle$,

C) $\langle T1, V1\rangle\alpha_0 + \langle T1, VP_2\rangle\alpha_1 + \langle T1, VP_4\rangle\alpha_2 = \langle 1, V1\rangle$,
$\langle T1, VP_2\rangle\alpha_0 + \langle TP_2, VP_2\rangle\alpha_1 + \langle TP_2, VP_4\rangle\alpha_2 = \langle 1, VP_2\rangle$,
$\langle T1, VP_4\rangle\alpha_0 + \langle TP_2, VP_4\rangle\alpha_1 + \langle TP_4, VP_4\rangle\alpha_2 = \langle 1, VP_4\rangle$.

In order to evaluate the functionals appearing in Eqs. (2.12)–(2.15), $P_n(x)$ is taken of the form

$$P_n(x) = \frac{1}{2^n} \sum_{\nu=0}^{[n/2]} \frac{(-1)^\nu (2n-2\nu)!}{\nu!(n-\nu)!(n-2\nu)!} x^{n-2\nu}.$$

Furthermore one has to take into account the following:

$$Tx^m = x^m + \frac{h}{m+1}\left[\left(\ln\frac{4}{R} - \gamma + \frac{m+2}{m+1}\right) - (1-x^{m+1})\ln(1-x) - x^{m+1}\ln x + \sum_{i=0}^{m} \frac{x^{m-i}}{1+i} - \frac{1}{m+1}\right].$$

Moreover, the relation below holds:

$$\langle TP_n(x), VP_m(x)\rangle = \langle VP_n(x), TP_m(x)\rangle.$$

3. Results

As shown, the drag coefficient C_D can be evaluated from the upper and lower bounds of the considered functionals. In order to apply the method in a significant situation, the case of a slightly rarefied flow of spherical molecules will be considered. In this circumstance the slip coefficient can be assumed to be equal to the Knudsen number and this, in turn, can be expressed by the ratio $\lambda S/R$ where $\lambda = 16/5\pi^{1/2}$ and S is the speed ratio, U/C_m, of the unperturbed velocity to the most probable molecular speed C_m.

Tables (1–2) show the calculated values of C_D as obtained via the three approximations of L_0, L_1, and U, at different values of R and S. Only the first approximation of L_2 was actually computed, due to the complexity of this particular functional.

Table 1. Drag coefficient for $S = 0.01$

$C_D = W(\phi f)$ / R	0.01	0.1	1.0	4.0
$L_0(\phi_0)$	130.51	37.256	7.34361	3.22045
$L_0(\phi_1)$	130.51	37.263	7.3491	3.2251
$L_0(\phi_2)$	130.52	37.356	7.3491	3.2251
$L_1(\phi_0)$	130.52	37.356	7.7311	4.4054
$L_1(\phi_1)$	130.52	37.364	7.7268	4.3874
$L_1(\phi_2)$	130.53	37.379	7.5404	3.5739
$L_2(\phi_0)$	130.52	37.290	7.3699	3.2570
$U(\phi_0)$	130.53	37.428	8.0081	5.2603
$U(\phi_1)$	130.53	37.426	7.9977	5.2350
$U(\phi_2)$	130.53	37.392	7.6141	3.7902
C_D(Ref. [1])	162.72	38.102	7.5090	3.3880
C_D(Ref. [4])	130.53	37.384	7.5104	3.3990

Table 2. Drag coefficient for $S = 0.16$

$C_D = W(\phi, f)$ / R	0.01	0.1	1.0	4.0
$L^5(\phi^5)$	13.269	10.577	4.9050	2.6440
$L^5(\phi_1)$	13.269	10.577	4.9057	2.6460
$L^5(\phi_2)$	13.269	10.578	4.9208	2.6983
$L_1(\phi_0)$	13.269	10.578	4.9158	2.6939
$L_1(\phi_1)$	13.269	10.578	4.9161	2.6939
$L_1(\phi_2)$	13.269	10.578	4.9206	2.6975
$L_2(\phi_0)$	13.269	10.578	4.9120	2.6623
$U(\phi_0)$	13.269	10.578	4.9235	2.7297
$U(\phi_1)$	13.269	10.578	4.9234	2.7288
$U(\phi_2)$	13.269	10.578	4.9208	2.6983
C_D(Ref. [1])	1982.621	8.724	4.6860	2.6080
C_D(Ref. [4])	13.269	10.578	4.9201	2.6918

Table 3. Relative errors, r

R / S	0.01	0.02	0.04	0.08	0.16
0.01	7.6×10^{-6}	1.2×10^{-6}	1.1×10^{-6}	10^{-7}	10^{-7}
0.1	1.8×10^{-4}	5×10^{-5}	1.2×10^{-5}	10^{-7}	10^{-7}
1.0	4.8×10^{-3}	1.09×10^{-3}	3.1×10^{-4}	8.3×10^{-5}	1.1×10^{-5}
4.0	2.9×10^{-2}	7.4×10^{-3}	3.3×10^{-2}	5.6×10^{-4}	1.4×10^{-4}

Finally, Table 3 shows the relative error defined by

$$r = \frac{U(\phi_2)-L_1(\phi_2)}{U(\phi_2)+L_1(\phi_2)}$$

for a number of values of the Reynolds number and of S.

Inspection of the results shows the excellent convergence characteristics of the method, convergence improving at lower R and at increasing S for each functional.

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In Tables (1–2) the drag coefficient evaluated according to the approximate theory of Tamada and Miura are reported together with the exact results obtained by solving Eq. (1.1) by an integral transform method [4].

As one can see, the agreement with the results of the much more difficult exact evaluation is excellent. On the contrary, the data obtained through the approximate theory of [1] are very good only at relatively high values of R, the accuracy decreasing with S.

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