Archives of Mechanics • Archiwum Mechaniki Stosowanej • 30, 4–5, pp. 595-601, Warszawa 1978

On spatial free-boundary flows

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THIS PAPER constitutes an effort to study the correctness of the boundary-value problem for a three-dimensional flow with free boundaries. The region of flow D is bounded by a surface composed of the surfaces S and Σ. The surface S is prescribed by the equation $F(\mathbf{x}) = 0$. The surface Σ is unknown and should be evaluated jointly with determination of the velocity $\mathbf{V}(\mathbf{x})$ in the region D. This is performed on the basis of the system of equations rot $V = 0$, div $\mathbf{V} = 0$ with boundary conditions $(\mathbf{V}\cdot\mathbf{n}) = 0$ for $\mathbf{x} \in S\cup\Sigma$, $|\mathbf{V}| = \lambda$ for $\mathbf{x} \in \Sigma$. The author has shown that generally the above problem is undetermined and depends of the shape of the surface S. At the sufficiently strong restrictions imposed on the surface S and applying the hodograph method the uniqueness of the previously formulated problem was demonstrated. It was also shown that in the certain particular three-dimensional cases the small local disturbances do not disappear but cause the rising of the system of surface waves.

Praca zawiera próbę zbadania poprawności problemu brzegowego dla trójwymiarowego przepływu z powierzchniami swobodnymi. Obszar przepływu D jest ograniczony powierzchnią składającą się z powierzchni S i Σ. Powierzchnia S jest dana równaniem $F(\mathbf{x}) = 0$. Powierzchnia Σ nie jest znana i powinna być wyznaczona w trakcie znajdowania prędkości $\mathbf{V}(\mathbf{x})$ w obszarze D. Dokonuje się tego na podstawie układu równań rot $V = 0$, div $\mathbf{V} = 0$ przy warunkach brzegowych $(\mathbf{V}\cdot\mathbf{n}) = 0$ dla $\mathbf{x} \in S\cup\Sigma$, $|\mathbf{V}| = \lambda$ dla $\mathbf{x} \in \Sigma$. Autor wykazał, że powyższy problem w ogólnym przypadku jest niedookreślony. Zależy to od kształtu powierzchni S. Przy dosyć silnych ograniczeniach, dotyczących powierzchni S, stosując metodę hodografu udowodniono jednoznaczność wyżej sformułowanego problemu. Również pokazano, że w pewnych szczególnych trójwymiarowych przypadkach małe lokalne zaburzenia nie znikają, lecz powodują powstawanie układu fal powierzchniowych.

Работа содержит попытку исследования корректности краевой задачи для трехмерного течения со свободными поверхностями. Область течения D ограничена поверхностью, состоящей из поверхностей S и Σ. Поверхность S задана уравнением $F(\mathbf{x}) = 0$. Поверхность Σ неизвестна и должна быть определена в процессе нахождения скорости $\mathbf{V}(\mathbf{x})$ в области D. Это производится на основе системы уравнений rot$\mathbf{V} = 0$, div$\mathbf{V} = 0$, при граничных условиях $(\mathbf{V}\cdot\mathbf{n}) = 0$ для $\mathbf{x} \in S\cup\Sigma$, $|\mathbf{V}| = \lambda$ для $\mathbf{x} \in \Sigma$. Автор показал, что выше приведенная задача в общем случае недоопределенная. Это зависит от формы поверхности S. При довольно сильных огрeничениях, касающихся поверхности S, применяя метод годографа, доказана единственность вышесформулированной задачи. Показано тоже, что в некоторых частных трехмерных случаях малые локальные возмущения не затухают, но вызывают возникновение системы поверхностных волн.

1. Introduction

THE THEORY of spatial free boundary flows is one of the least developed fields of hydrodynamics. At the present time it is not known whether the problem of a spatial free-boundary flow is mathematically well-posed [1, 2]. The given paper is the first attempt to decide this question.

The purpose of our study is to investigate the correctness of problems of steady spatial potential free boundary flows. In these problems the flow region D is in the space of points $\mathbf{x} = (x_1, x_2, x_3)$. The boundary of D consists of two components, S and Σ. The surface

S is given by the equations $F(\mathbf{x}) = 0$, the surface Σ is unknown. It is necessary to find Σ and the velocity field $\mathbf{V}(\mathbf{x})$ so that the equations

$$D: \operatorname{rot} \mathbf{V} = 0, \quad \operatorname{div} \mathbf{V} = 0; \tag{1.1}$$

$$S \cup \Sigma: (\mathbf{V}, \mathbf{n}) = 0; \tag{1.2}$$

$$\Sigma: |\mathbf{V}| = \lambda \tag{1.3}$$

are satisfied. Here $\mathbf{n}$ is the normal vector to the boundary of D, λ — some parameter.

There is a difference of principle between three- and two-dimensional free boundary problems. We illustrate it with a simple example: Let S be the plane $x_3 = 0$. In the two-dimensional case the uniform flow turns out to be a unique solution of Eqs. (1.1) – (1.3), and the surface Σ coincides with the plane $x_3 = \text{const}$. In the three-dimensional case the problem (1.1) – (1.3) has an infinite set of solutions:

$$\mathbf{V} = (\lambda, 0, 0), \quad \Sigma: x_3 = \sigma(x_2), \tag{1.4}$$

where σ is an arbitrary continuous function.

This suggests, generally speaking, that the problem (1.1) – (1.3) is sub-definite. In this work it will be shown that the correct formulation of the three-dimensional free boundary problem depends on the flow region geometry.

2. Hodograph method

Since the domain D is unknown it is convenient to use the hodograph method.

Let u denote the flow potential and $\mathbf{V}(\mathbf{x}_0) \neq 0$. It is well-known that in the neighbourhood of $\mathbf{x}_0$ there are the functions v, w which satisfy the equality

$$\nabla u - \nabla v \times \nabla w = 0. \tag{2.1}$$

The surface of the level v, w consist of streamlines, i.e. v, w are the stream functions. The relations (2.1) can be considered as the system of differential equations for the vector function $\mathbf{w} = (u, v, w)$.

Harmonic mappings are the solutions of Eq. (2.1). The existence and uniqueness theorems for the mappings of the layer-type regions to a plane layer are valid. We formulate this theorem for the case of an infinite region.

Let us assume that D satisfies the following conditions:

a) the surface S is given by the equation $x_3 = f(x_1, x_2)$ and asymptotically tends to the planes $x_3 = 0$ and $x_3 = x_1 \operatorname{tg}\beta$ if $x_1 \to \pm\infty$:

$$\begin{aligned} |x_1^3 D^\alpha (f - x_1 \operatorname{tg}\beta)| \to 0 \quad &\text{if} \quad x_1 \to \infty, \\ |x_1^3 D^\alpha f| \to 0 \quad &\text{if} \quad x_1 \to -\infty; \end{aligned} \tag{2.2}$$

b) the surface Σ is given by the equation $x_1 = f + h$. The depth of flow $h(x_1, x_2)$ is the smooth positive function. It is supposed that it tends to the limited values of $h^\pm(x_2)$ when $x_1 \to \pm\infty$:

$$|x_1^3 D^\alpha (h - h^\pm)| \to 0 \quad \text{if} \quad x_1 \to \pm\infty; \tag{2.3}$$

c) the vector field $\mathbf{V}(\mathbf{x})$ exists satisfying Eqs. (1.1) and (1.2) such that

$$(2.4)\qquad \begin{aligned} &|x_1^3 D^\alpha(\mathbf{V}-j^\pm)| \to 0 \quad \text{if} \quad x_1 \to \pm\infty,\\ &C > (\mathbf{V}, j^-) > C^{-1} > 0. \end{aligned}$$

Here $j^- = (1,0,0)$, $j^+ = (\cos\beta, 0, \sin\beta)$ are the vectors tangential to the asymptotes of S, α — the multi-index, $|\alpha| \leqslant 3$.

Let Ω denote the layer in the point space $\mathbf{w}$ bounded by the planes $\Gamma_0: w = 0$ and $\Gamma_1: w = H$ where the mean flow depth is expressed by

$$H = \lim_{N\to\infty} \frac{1}{2N} \int_{-N}^{N} h^-(t)\,dt.$$

Let $v = v^-(x_2, x_3)$, $w = w^-(x_2, x_3)$ be the arbitrary smooth mapping of the two-dimensional region $0 < x_3 < h^-(x_2)$ onto the region $0 < w < H$

$$\frac{\partial(v^-, w^-)}{\partial(x_2, x_3)} = 1.$$

For example,

$$v^- = \int_0^{x_2} h^-(t)\,dt, \qquad w^- = H\frac{x_3}{h^-(x_2)}.$$

Then we have the following theorem [3]:

THEOREM 1. *Under the above hypothesis the system of* (2.1) *has a unique solution satisfying the conditions*

$$v(\mathbf{x}) \to v^-(x_2, x_3), \qquad w(\mathbf{x}) \to w^-(x_2, x_3),$$
$$u(\mathbf{x}) - x_1 \to 0 \quad \text{if} \quad x_1 \to -\infty.$$

The vector function $\mathbf{w}(\mathbf{x})$ *is the diffeomorphism of* D *onto* Ω *with uniformly bounded derivatives.*

An analogous statement for a periodical case is proved in [4].

Let u, v, w be independent variables. In order to reduce the free boundary problem to the boundary value problem in the domain Ω for the vector function $\mathbf{x}(\mathbf{w})$, we turn to the reverse functions Eqs. (1.1) – (1.3). The equations

$$(2.5)\qquad \begin{aligned} &\Omega:\ \mathbf{x}_u - \mathbf{x}_v \times \mathbf{x}_w = 0;\\ &\Gamma_0:\ F(\mathbf{x}) = 0, \quad \Gamma_1: |\mathbf{x}_u| = \lambda^{-1} \end{aligned}$$

are obtained.

Every schlicht solution of the nonlinear boundary value problem (2.5) gives the solution of the free boundary problem (1.1) – (1.3).

Two-dimensional flows. If F does not depend on variables x_2, then

$$(2.6)\qquad \frac{\partial x_1}{\partial v} = \frac{\partial x_3}{\partial v} = 0, \qquad x_2 = v.$$

In this case Eqs. (2.5) turn into the Cauchy-Riemann system for the functions $x_1(u, w)$, $x_3(u, w)$.

3. Existence theorem

Let a certain approximate solution $\mathbf{V}_0$ of the problem (1.1) – (1.3) with the surface S_0 be given, and the approximate solution $\mathbf{x}_0(\mathbf{w})$ of Eqs. (2.5) correspond to it. We shall find the exact solution of the problem (1.1) – (1.3) with the given surface S which is close to S_0. Now it is necessary to find an exact solution of the problem (2.5) which is close to $\mathbf{x}_0(\mathbf{w})$.

The new required vector function $\mathbf{z}$ is introduced by the equality $\mathbf{x}_0(\mathbf{z}) = \mathbf{x}$. The identical mapping $\mathbf{z}_0(\mathbf{w}) = \mathbf{w}$ corresponds to the initial solution $\mathbf{x}_0$. After a change of variables we shall obtain the following equations for $\mathbf{z}(\mathbf{w})$:

$$\begin{aligned} &\Omega: L(\mathbf{z}) \equiv (\det G)^{-\frac{1}{2}} G\mathbf{z}_u - \mathbf{z}_v \times \mathbf{z}_w = 0;\\ &\Gamma_1:\ B(\mathbf{z}) \equiv (G\mathbf{z}_u, \mathbf{z}_u) = \lambda^{-1};\\ &\Gamma_0: z_3 - \varepsilon\Phi(\mathbf{z}) = 0. \end{aligned} \tag{3.1}$$

Here Φ is the function giving the perturbation of the surface S_0, G is the matrix with the elements

$$G_{ij} = \left(\frac{\partial \mathbf{x}_0}{\partial z_i}(\mathbf{z}),\ \frac{\partial \mathbf{x}_0}{\partial z_j}(\mathbf{z})\right).$$

It is assumed that: a) the matrix G is diagonal and its elements satisfy the inequalities $G_{ii} > \alpha > 0$. The functions $G_{ii}(\mathbf{z})$, $\Phi(\mathbf{z})$ are periodical with respect to z_1, z_2 and belong to the space $C_5(D_0)$. D_0 is the region $-\alpha < z_3 < \alpha + H$; b) the identical mapping $\mathbf{z}_0$ is an approximate solution of Eqs. (3.1):

$$||L(\mathbf{z}_0)||_{C_5(\Omega)} + ||\Phi(\mathbf{z}_0)||_{C_5(\Omega)} + ||\lambda_0 B(\mathbf{z}_0) - 1||_{C_5(\Gamma_1)} < \varepsilon;$$

c) the matrix G satisfies the conditions

$$\frac{\partial B(\mathbf{z}_0)}{\partial w} > \alpha > 0 \quad \text{if} \quad w = H. \tag{3.2}$$

Here α, λ_0 are some positive constants, ε is a small parameter. The following theorem is valid:

THEOREM 2. *Under the above conditions and for sufficiently small positive ε the problem* (3.1) *has at least one solution*:

$$\mathbf{z}(\mathbf{w}) = \mathbf{w} + \mathbf{Z}(\mathbf{w}).$$

The *vector function $\mathbf{z}$ diffeomorphically maps Ω onto a certain region $D \subset D_0$. The vector function $\mathbf{Z}$ is periodical with respect to u, v and $||Z||_{C_2(\Omega)} \to 0$ if $\varepsilon \to 0$.*

As an application of this theorem, we discuss the following problem concerning flows on a torus: Let S_j, $j = 0, 1$ be two coaxial tori in R^3 which are given by the equations

$$(\varepsilon|\mathbf{x}|^2 + 1 - \varepsilon r_j^2)^2 = 4(|\mathbf{x}|^2 - x_3)^2,$$

$$r_0 = 1, \quad r_1 = \exp H.$$

The configuration $S = S_0$, $\Sigma = S_1$ forms an approximate solution of the free boundary problem (1.1) – (1.3). The mapping

$$x_{1,0} = Q\frac{\cos\varepsilon v}{\varepsilon}, \quad x_{2,0} = Q\frac{\sin\varepsilon v}{\varepsilon}, \quad x_{3,0} = e^w \sin u,$$

$$Q = 1 - \varepsilon e^w \cos u$$

corresponds to it. After turning to the variables **z** we shall obtain the problem (3.1) with the diagonal matrix G. Its elements

$$G_{22} = Q^2, \quad G_{11} = G_{33} = e^{2z_3}$$

satisfy all the conditions of Theorem 2. It follows from this that the problem (1.1) – (1.3) is solvable in the case when the given surface S is close to torus S_0, and ε is sufficiently small.

The inequality (3.2) is the main restriction to the class of flows under consideration. It means that on a free surface the pressure gradient vector is directed "from fluid".

4. The linear model

The transition to the hodograph variables makes it possible to introduce simple approximate equations describing flows which are close to plane ones. Consider the problem of an imponderable free boundary flow above a rough bottom.

Let $g(x_1, x_2)$ be a smooth finite function. Assume that $\lambda = 1$, the surface S is given by the equation

$$x_3 = f_0(x_1) + \varepsilon g(x_1, x_2) \tag{4.1}$$

and satisfies the condition (2.2). It is necessary to find the solution of the problem (1.1) – (1.3) with the velocity field **V** and the stream depth h satisfying the conditions (2.3) – (2.4). The functions $h^{\pm}$ are unknown.

For $\varepsilon = 0$ the problem has a unique solution $\mathbf{V}_0(x_1, x_3)$, $h_0(x_1)$ in the class of two-dimensional flows, [2, 6, 7], and the functions $h^{\pm}$ are constant:

$$h^- = h^+ \cos\beta = H. \tag{4.2}$$

Example (1.4) shows that in the three-dimensional case the problem (1.1) – (1.3), (2.3) – (2.4) is sub-defined. In [2] the hypothesis is suggested that for the correctness of the problem (1.1) – (1.3) it is necessary to demand that the conditions (4.2) be satisfied.

We will show, whithin the framework of an approximate model, that the correct formulation of the spatial free boundary problem depends on the flow region geometry. For some flows the conditions (1.1) – (1.3), (2.3) – (2.4) are sufficient for the existence and uniqueness of the solution. The condition (4.2) is not satisfied in this case.

Derivation of approximate equations. Let us consider the problem (1.1) – (1.3), (2.3) – (2.4) in hodograph variables. Let $\mathbf{x}_0(\mathbf{w})$ be the solution (2.5) corresponding to $\mathbf{V}_0$, h_0. According to Theorem 1, such a solution exists and satisfies Eqs. (4.2) and (2.4).

Denote the new vector function with the components φ, ψ_1, ψ_2 by $\boldsymbol{\varphi}$, Let $\psi(u, v)$ be the restriction ψ_2 to the plane Γ_1, M — the Jacoby matrix of the mapping $\mathbf{x}_0(\mathbf{w})$. We will find the solution (2.5) in the form

$$\mathbf{x} = \mathbf{x}_0 + \varepsilon M^* \boldsymbol{\varphi}. \tag{4.3}$$

It follows from this that the formula for the disturbed flow depth h has the form

$$h - h_0 - \varepsilon g = \varepsilon\psi(s, x_2)\cos\theta(s) + O(\varepsilon^2),$$
$$h^- - H = \varepsilon\psi^-(x_2), \quad h^+ - H = \varepsilon\psi^+(x_2)\operatorname{cosec}\beta.$$

Here s is the arc abscissa of the curve $x_3 = f_0 + h_0$, θ the angle between the tangent of this curve and the axis of the abscissa,

$$\psi^{\pm}(v) = \lim_{u \to \pm\infty} \psi(u, v).$$

It follows that ψ coincides with the value of the free surface disturbance. If we substitute Eq. (4.3) into Eq. (2.5), reject terms in the order of ε^2 and axclude ψ_1, ψ_2, we will obtain the following system of equations for φ, ψ:

$$\begin{aligned} &\Omega: \operatorname{div}(A\nabla\varphi) = 0; \\ &\Gamma_1: \varphi_u - a\psi = 0, \quad \psi_u + \varphi_w = 0; \\ &\Gamma_0: \varphi_w = g_0. \end{aligned} \tag{4.4}$$

The function g_0 depends only on g (1.3). The element of the diagonal matrix A and the coefficient a are in the form

$$A_{11} = A_{33} = 1, \quad A_{22} = |\mathbf{x}_{0.u}|^2, \quad a = \frac{\partial}{\partial u}\theta(u).$$

The conditions at infinity can be written as follows:

$$\begin{aligned} &\varphi \to 0 \quad \text{if} \quad u \to -\infty, \quad |\varphi_u| \to 0 \quad \text{if} \quad |u| \to \infty, \\ &\lim_{N\to\infty} \frac{1}{2N} \int_{-N}^{N} \psi^-(t)\,dt = 0. \end{aligned} \tag{4.5}$$

The additional condition (4.2), within the framework of a linear model, is in the form

$$\psi^{\pm}(v) = 0. \tag{4.6}$$

Equations (4.4) form a boundary value problem for φ and the free surface disturbance. In the two-dimensional case the problem (4.4) coincides with the boundary value problem for the Veinstein function (2.2), (2.3).

For a wide class of flows the conditions (4.6) and (4.5) provide a uniqueness of the solution (4.4). Assume that the following conditions are valid.

The function a vanishes only in a finite number of points u_i, $i = 1, \ldots, k$, $k \leqslant 2$. If $k = 2$, then within the interval between zeros u_1, u_2 the function a is negative. The matrix A and the coefficient a satisfy the inequalities

$$\begin{aligned} &a_u(u) > 0, \quad \varrho_0(u)|a(u)| \geqslant m\lambda \quad \text{if} \quad |u| > T; \\ &|a_u(u_i)| \geqslant m\lambda, \quad \|\varrho_0 a\|_{C_2(R^1)} \leqslant \lambda. \end{aligned}$$

Here T, m, λ are the positive constants, $\varrho_0 = (1+u^2)^\sigma$, $\sigma > 1$ the weight function.

The following theorem [3], which is a spatial analogy of Friedrichs-Veinstein lemma [6] is valid.

THEOREM 3. *Let φ, ψ be the solution* (4.4)–(4.5) *with* $g_0 = 0$, $\nabla\varphi \in L_2(\Omega)$ *satisfying the condition*

$$\sum_{|\alpha| \leqslant 2} \int_{\Gamma_1} |D^\alpha \psi|^2 du\,dv < \infty.$$

Then the positive constant $\lambda_0 = \lambda_0\ (m, T)$ *exists such that if* $\lambda < \lambda_0$ *then* $\psi = \varphi = 0$.

For some classes of flows the additional condition (4.6) is unnecessary. The problem (4.4)–(4.5) is correct in the class of bounded functions. Thus the following theorem is valid:

THEOREM 4. *Let the function a satisfy the inequality*

$$m^{-1}\varrho_0 < a < m\varrho.$$

For any finite $g_0 \in C_3(\Gamma_1)$ *the problem* (4.5)–(4.4) *has the unique solution* $(\varphi, \psi) \in C_2(\Omega) \times C_2(\Gamma_1)$ *for which the estimate*

$$\|\sqrt{\varrho_0}\nabla\psi\|_{L_2(\Gamma_1)} + \|\psi^{\pm}\|_{L_2(-\infty,\infty)} \leqslant C \sum_{|\alpha|\leqslant 2} \|\varrho_0 D^{\alpha} g_0\|_{L_2(\Gamma_1)},$$

is valid. The functions $\psi^{\pm}$ *are equal to zero if and only if* $g_0 = 0$.

Thus it has been proved that in some cases the local disturbances of a spatial flow do not vanish at infinity, but lead to the appearance of the system of surface waves.

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Received September 27, 1977.